MATH 54 – FINAL EXAM

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Name: ____

Instructions: This is it, guys! Your final hurdle to freedom! :) This final counts for 20% of your grade and you officially have 110 minutes to take this exam. There are lots of questions, so don't spend *too* much time on each question! Prepare for the final battle!!!

1		10
2		10
3		10
4		10
5		35
6		5
7		10
8		5
9		3
10		2
Bonus		1
Total		100

Date: Friday, August 10th, 2012.

1. *(10 points, 2 points each)*

Label the following statements as **T** or **F**. **Write your answers in the box below!**

NOTE: In this question, you do **NOT** have to show your work! Don't spend *too* much time on each question!

(a) If $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{x}$ on W, then $\hat{\mathbf{x}}$ is orthogonal to $\mathbf{x}$.

(b) If $\hat{\mathbf{u}}$ is the orthogonal projection of $\mathbf{u}$ on $Span\{\mathbf{v}\}$, then:

$$\hat{\mathbf{u}} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{u}$$

(c) For any (continuous) f and g,

$$\left(\int_0^1 f(t)g(t)dt\right)^2 \leq \left(\int_0^1 (f(t))^2 dt\right)\left(\int_0^1 (g(t))^2 dt\right)$$

(d) If $\hat{\mathbf{x}}$ is the least-squares solution of $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, then $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{x}$ on $Col(A)$

(e) If Q is an orthogonal matrix, then Q is invertible.

(a)	
(b)	
(c)	
(d)	
(e)	

2. *(10 points)* Apply the Gram-Schmidt process to find an *orthonormal* basis of W, where:

$$W = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

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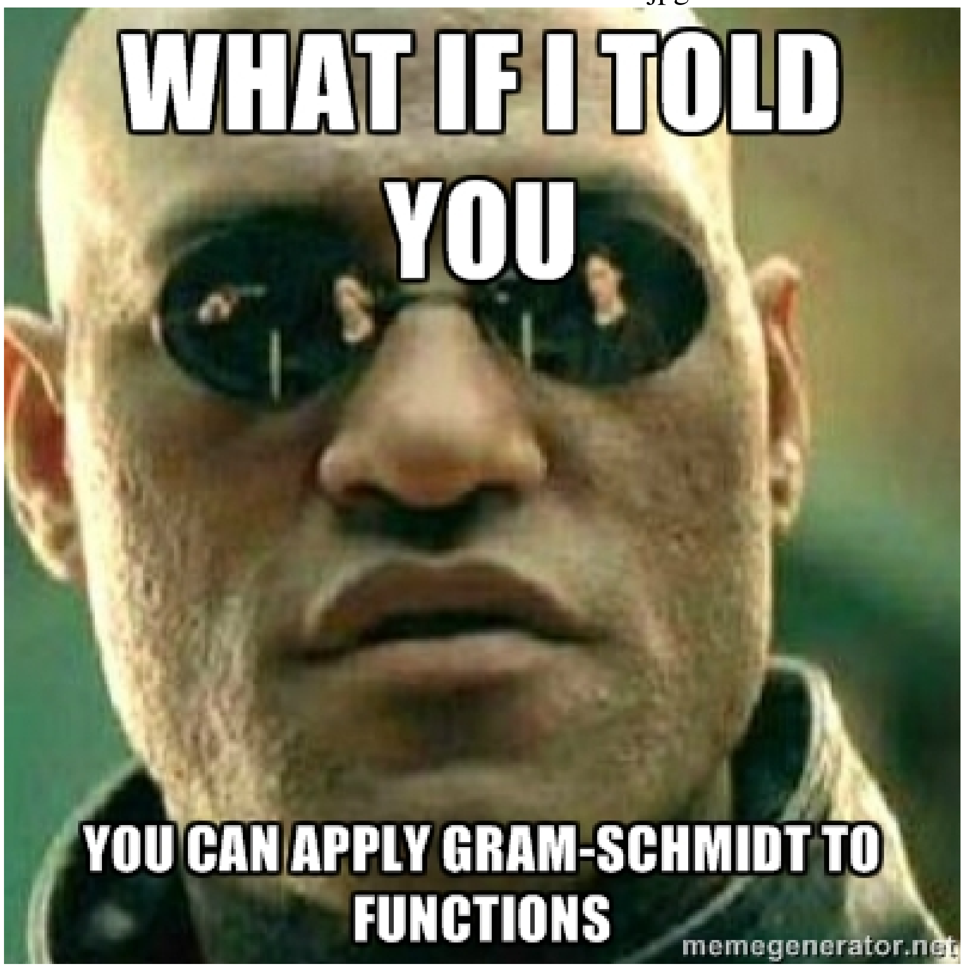

3. *(10 points)* Consider the space $C[-\frac{\pi}{2}, \frac{\pi}{2}]$ with the dot product:

$$f \cdot g = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)g(t)dt$$

Find the orthogonal projection of $f(x) = \cos(x)$ on

$$W = Span\{1, \sin(x), \sin(2x)\}$$

And use this to find a function g which is orthogonal to f.

Note: Don't waste *too* much time calculating the integrals, this should be quicker than you think!

4. *(10 points)* Consider the (inconsistent) system of equations $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

(a) *(5 points)* Find the orthogonal projection of $\mathbf{b}$ on $Col(A)$

Hint: The columns of A are orthogonal!

(b) *(5 points)* Use your answer in (a) to find a least-squares solution to the system $A\mathbf{x} = \mathbf{b}$

Note: If you're completely stuck with (a) and (b), just find the least-squares solution the usual way (for a maximum of 5 points out of 10)

5. *(35 points)* Find a solution to the following wave equation:

$$\begin{cases} u_{tt} = 9u_{xx} & 0 < x < \pi, \quad t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = x^2(\pi - x) & 0 < x < \pi \\ u_t(x,0) = 0 & 0 < x < \pi \end{cases}$$

Note: Make sure to show *all* your work, and make sure to do this problem from scratch. Also, at some point, you may have an integral on the denominator. That integral is equal to π. Finally, be careful!

(Scratch work)

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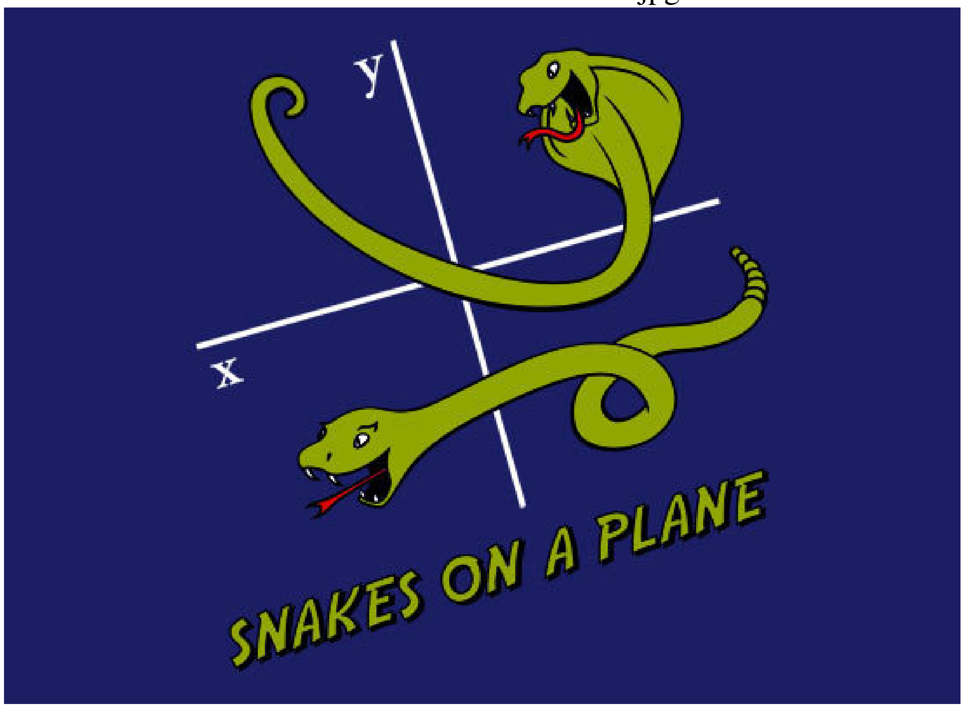

6. *(5 points)* Consider $f(x) = x^2 + 1$ on $(0, 1)$.

Draw the graph of $\mathcal{F}(x)$, the Fourier *sine* series of f on $(-4, 4)$ Make sure to label what happens at the endpoints!

7. *(10 points)* Consider $f(x) = \begin{cases} 0 & \text{on}(-1,0) \\ 1 & \text{on}(0,1) \end{cases}.$

Parseval's identity states that:

$$\sum_{m=0}^{\infty} (A_m)^2 + (B_m)^2 = \int_{-1}^{1} (f(x))^2$$

Where A_m and B_m are the (full) Fourier coefficients of f.

Calculate A_m and B_m and use this to calculate:

$$\sum_{m=1, m \; odd}^{\infty} \frac{1}{m^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} \cdots$$

Hint: Think of a **Pey**a**m**-**ex**pansion! Also, at some point, you may have an integral on the denominator. That integral is equal to 1.

8. *(5 points)* Use the following steps to give an alternate and easier proof of the Cauchy-Schwarz inequality. All the questions are pretty much independent (except for (d))

(a) *(1 point)* What does the Cauchy-Schwarz inequality say?

(b) *(1 point)* What is the formula of $\hat{\mathbf{u}}$, the projection of $\mathbf{u}$ on $Span\{\mathbf{v}\}$?

(c) *(1 point)* Circle the correct answer:

(A) $\|\hat{\mathbf{u}}\| \leq \|\mathbf{u}\|$

(B) $\|\mathbf{u}\| \leq \|\hat{\mathbf{u}}\|$

(d) *(2 points)* Use your formula in (b) and your answer in (c) to solve for $\mathbf{u} \cdot \mathbf{v}$ and (hence) derive the Cauchy-Schwarz inequality!

Note: Be careful about when to put $|\cdot|$ or $\|\cdot\|$.

9. *(3 points)* Suppose $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthonormal. Show that $\mathcal{B}$ is linearly independent!

Hint: Use hugging!

Note: Let me start the proof for you:

Suppose $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$.

Goal: Show that $a = b = c = 0$

10. *(2 points)* Who's your favorite Math 54 teacher of all time??? :D Any other good-bye words?

Thank you for flying Peyam Airlines! I hope you had a safe and enjoyable Math 54 - trip, and I hope to see you on board again soon! :)

Bonus *(1 point)* Find the general solution to the following PDE:

$$\begin{cases} u_{xx} + u_{yy} = u \\ u(0,y) = u(1,y) = 0 \end{cases}$$

(where $u = u(x,y)$ and $0 < x < 1, 0 < y < 1$)

Congratulations!!!

You're done with Math 54! :D

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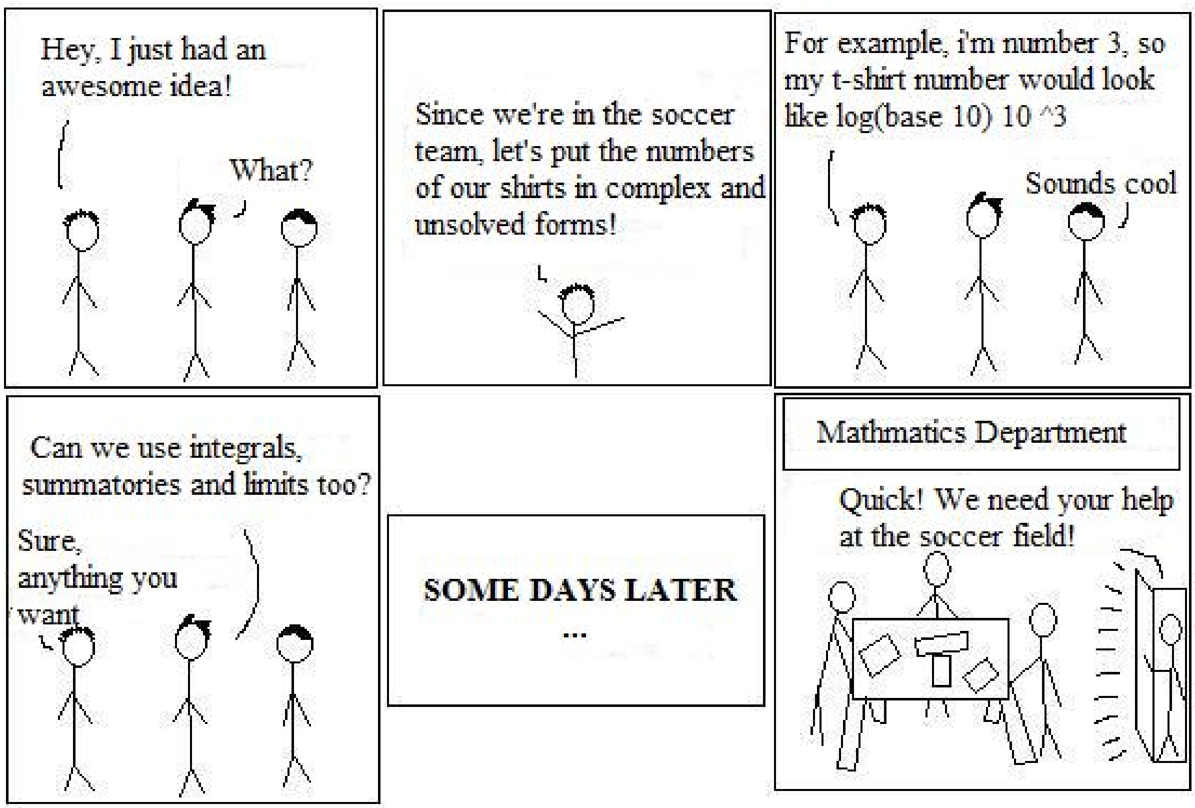

(Scratch work)

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